

Hypergeometric representation of a four-loop vacuum bubble

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In this article, we present a new analytic result for a certain single-mass-scale four-loop vacuum (bubble) integral. We also discuss its systematic ϵ -expansion in $d = 4 - 2\epsilon$ as well as $d = 3 - 2\epsilon$ dimensions, the coefficients of which are expressible in terms of harmonic sums at infinity.

1. Introduction

Recently, a number of calculations have reached the four-loop level [1]. In most cases, such high-loop computations proceed by first reducing the (typically very large number of) momentum-space integrals to a small number of so-called master integrals, a step that can be highly automatized [2]. In a second step, these master integrals have to be computed in an ϵ -expansion around the space-time dimension d of interest.

This second step is much less amenable to automatization, at least if one is interested in analytic expressions for the coefficients. Indeed, very few four-loop master integrals are known analytically. It is the purpose of this paper to add one master integral to this set.

As a basic building block, many of the above-mentioned calculations use single-mass-scale vacuum bubbles. These arise e.g. when there are large scale hierarchies in the physics problem, such that asymptotic expansions can be used. Having no external momenta, and propagators with masses 0 or m only, members of this class of integrals are functions of the space-time dimension d only (for convenience, we set $m = 1$ in the remainder of this paper), hence the coefficients of their Laurent expansions in ϵ are pure numbers.

Define the (dimensionless, but d -dependent) 1-, 3- and 4-loop integrals

$$J(x) \equiv \int_p^{(d)} \frac{1}{(p^2+1)^x},$$

$$B_2(x) \equiv \int_{p_{1,2,3}}^{(d)} \frac{1}{(p_1^2+1)^x} \frac{1}{p_2^2+1} \frac{1}{p_3^2} \frac{1}{(p_1+p_2+p_3)^2}$$

$$T_3(x) \equiv \int_{p_{1,2,3,4}}^{(d)} \frac{1}{(p_1^2+1)^x} \frac{1}{(p_1+p_4)^2} \frac{1}{p_2^2+1} \times \\ \times \frac{1}{(p_2+p_4)^2} \frac{1}{p_3^2+1} \frac{1}{(p_3+p_4)^2}.$$

At $x = 1$, all three examples are master integrals. While $J(1)$ and $B_2(1)$ can be computed analytically in terms of Gamma functions, we would like to know $T_3(1)$.

2. Difference equation

Using a slight generalization of the abovementioned reduction step, one can systematically derive sets of difference equations for the integrals. For an integral $f(x)$, they have the generic form $\sum_{i=0}^N c_i f(x+i) = g(x, d)$, with polynomial coefficients $c_i(d)$, and where N is the *order* of the difference equation and $g(x)$ its inhomogeneity, which is supposed to be known analytically.

First-order difference equations can be solved directly in terms of one initial value.

In the homogeneous case, this can be done trivially in terms of Pochhammer symbols $(a)_x \equiv \Gamma(a+x)/\Gamma(a)$:

$$J(x+1) = \frac{x - \frac{d}{2}}{x} J(x) \\ = \frac{(1 - \frac{d}{2})_x}{(1)_x} J(1), \quad (1)$$

$$B_2(x+1) = \frac{(x+2-d)(x+3-\frac{3d}{2})}{x(x+5-2d)} B_2(x) \\ = \frac{(3-d)_x (4-\frac{3d}{2})_x}{(1)_x (6-2d)_x} B_2(1). \quad (2)$$

For these two cases, we know the initial values $J(1)$ and $B_2(1)$ analytically in d dimensions. The value of $J(1)$ depends on the choice of integration measure, which we do not need to specify here¹, and a simple integration gives

$$B_2(1) = \frac{2^{d-3} \Gamma(4 - \frac{3d}{2}) \Gamma(\frac{3}{2} - \frac{d}{2}) \Gamma(\frac{d}{2})}{\Gamma(\frac{7}{2} - d) \Gamma(1 - \frac{d}{2})} [J(1)]^3 \quad (3)$$

The 4-loop integral T_3 is an example for the inhomogeneous case,

$$\begin{aligned} T_3(x+1) &= \frac{(x+2-d)(x+5-2d)}{x(x+8-3d)} T_3(x) + \\ &+ \frac{1}{x+8-3d} \left[(d-2)J(1)B_2(x+1) + \right. \\ &\left. + \left(4 - \frac{3d}{2}\right) B_2(1)J(x+1) \right] \quad (4) \end{aligned}$$

$$\equiv c(x)T_3(x) + G(x). \quad (5)$$

This difference equation has been solved numerically in 3d [3] and 4d [4]. We will solve this equation analytically in the following section.

3. Solution

One can immediately write down a solution of Eq. (5) in terms of a sum, as can be easily seen by doing the first couple of iterations,

$$\begin{aligned} T_3(x+1) &= T_3(x_0) \left(\prod_{i=x_0}^x c(i) \right) + \\ &+ \sum_{j=x_0}^x G(j) \left(\prod_{i=j+1}^x c(i) \right). \quad (6) \end{aligned}$$

In this case however, it is the initial value $T_3(1)$ that we would like to compute. What we instead know about the integral $T_3(x)$ is its behavior at the boundary, $T_3(x \gg 1) \propto J(x \gg 1)$.

To proceed, let us rewrite Eq. (6) as

$$\begin{aligned} T_3(x_0) &= T_3(x+1) \left(\prod_{i=x_0}^x \frac{1}{c(i)} \right) - \\ &- \sum_{j=x_0}^x G(j) \left(\prod_{i=x_0}^j \frac{1}{c(i)} \right) \quad (7) \end{aligned}$$

¹To display one particular choice, $J(1) = \int d^d k / (k^2 + 1) = \pi^{d/2} \Gamma(1 - d/2)$.

$$\begin{aligned} &= \left[T_3(x+1) \frac{\Gamma(x+1)\Gamma(x+9-3d)}{\Gamma(x+3-d)\Gamma(x+6-2d)} - \right. \\ &- \sum_{j=x_0}^x G(j) \frac{\Gamma(j+1)\Gamma(j+9-3d)}{\Gamma(j+3-d)\Gamma(j+6-2d)} \left. \right] \times \\ &\times \frac{\Gamma(x_0+2-d)\Gamma(x_0+5-2d)}{\Gamma(x_0)\Gamma(x_0+8-3d)}. \quad (8) \end{aligned}$$

The left-hand side does not depend on x , so the same has to be true for the right-hand side. Hence, we can freely choose the value of x at which we want to evaluate the right-hand side; in particular we can choose $x \gg 1$ or even perform the limit $\lim_{x \rightarrow \infty}$.

Using Stirling's formula $\Gamma(x+a)/\Gamma(x+b) \stackrel{x \gg 1}{\sim} x^{a-b} (1 + \mathcal{O}(x^{-1}))$, we see that the first term on the right-hand side of Eq. (8) behaves like $x^{1-d/2}$ at large x , so it vanishes for $d > 2$. So we are left to evaluate

$$\begin{aligned} T_3(x_0) &\stackrel{d \geq 2}{=} - \frac{\Gamma(x_0+2-d)\Gamma(x_0+5-2d)}{\Gamma(x_0)\Gamma(x_0+8-3d)} \times \\ &\times \sum_{j=0}^{\infty} \frac{\Gamma(j+x_0+1)\Gamma(j+x_0+8-3d)}{\Gamma(j+x_0+3-d)\Gamma(j+x_0+6-2d)} \times \\ &\times \left[(d-2) \frac{B_2(j+x_0+1)}{B_2(1)} + \right. \\ &\left. + \frac{8-3d}{2} \frac{J(j+x_0+1)}{J(1)} \right] J(1)B_2(1). \quad (9) \end{aligned}$$

Multiplying by $(1)_j/(1)_j$, the sum is just a generalized hypergeometric function ${}_J F_J$ at unit argument,

$$\begin{aligned} T_3(x_0) &\stackrel{d \geq 2}{=} \frac{\Gamma(x_0+2-d)(4 - \frac{3d}{2})}{(x_0+5-2d)\Gamma(x_0)\Gamma(2-d)} J(1)B_2(1) \times \\ &\times \left[{}_3F_2(i_1, i_2, 1; i_3, i_3; 1) \frac{\Gamma(i_1)\Gamma(6-2d)}{\Gamma(5 - \frac{3d}{2})\Gamma(i_3)} - \right. \\ &- {}_3F_2(i_4, i_2, 1; i_5, i_3; 1) \frac{\Gamma(i_4)\Gamma(2-d)}{\Gamma(1 - \frac{d}{2})\Gamma(i_5)} \left. \right] \end{aligned}$$

where $i_1 = x_0 + 4 - \frac{3d}{2}$, $i_2 = x_0 + 8 - 3d$, $i_3 = x_0 + 6 - 2d$, $i_4 = x_0 + 1 - \frac{d}{2}$ and $i_5 = x_0 + 3 - d$.

From this, setting $x_0 = 1$, we finally get [5]

$$T_3(1) \stackrel{d \geq 2}{=} \frac{(2-d)(8-3d)}{8(3-d)^2} J(1)B_2(1) \times$$

XSummer cannot expand the ${}_3F_2$ around half-integer indices, so unfortunately we cannot proceed the same way as above. There are algorithms for expansion around rationals p/q [12], which work only if rationals are *balanced* between numerator and denominator. The case at hand is *unbalanced*.

To balance it, we may make use of the Euler identity for Gauß' hypergeometric function

$${}_2F_1(\alpha, \beta; \delta; x) = (1-x)^{\delta-\alpha-\beta} {}_2F_1(\delta-\alpha, \delta-\beta; \delta; x)$$

which allows us to rewrite

$$\begin{aligned} {}_3F_2(\alpha, \beta, \gamma; \delta, \epsilon; 1) &= \\ &= \frac{\Gamma(\epsilon)}{\Gamma(\gamma)\Gamma(\epsilon-\gamma)} \int_0^1 dx \frac{(1-x)^{\epsilon-\gamma-1}}{x^{1-\gamma}} {}_2F_1(\alpha, \beta; \delta; x) \\ &= \frac{\Gamma(\epsilon)\Gamma(\epsilon+\delta-\alpha-\beta-\gamma)}{\Gamma(\epsilon-\gamma)\Gamma(\epsilon+\delta-\alpha-\beta)} \times \\ &\times {}_3F_2(\delta-\alpha, \delta-\beta, \gamma; \delta, \epsilon+\delta-\alpha-\beta; 1). \end{aligned} \quad (15)$$

This gives 6 transforms, 4 of which are *balanced*. Choosing one of those possibilities²,

$$\begin{aligned} {}_3F_2\left(1, \frac{1}{2} + (a-1)\epsilon, 6\epsilon; 1+a\epsilon, 1+4\epsilon; 1\right) &= \\ &= \frac{\Gamma(1+4\epsilon)\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\frac{1}{2}+5\epsilon)} \times \\ &\times {}_3F_2\left(a\epsilon, \frac{1}{2} + \epsilon, 6\epsilon; 1+a\epsilon, \frac{1}{2} + 5\epsilon; 1\right) \\ &= \frac{\Gamma(1+4\epsilon)\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(\frac{1}{2}+5\epsilon)} \times \\ &\times \left(1 + \sum_{i=1}^{\infty} \frac{(a\epsilon)_i (\frac{1}{2} + \epsilon)_i (6\epsilon)_i}{(1+a\epsilon)_i (\frac{1}{2} + 5\epsilon)_i (1)_i}\right), \end{aligned} \quad (16)$$

where $(n)_i = \frac{\Gamma(i+n)}{\Gamma(n)}$ are Pochhammer symbols. The sum is of “type A” according to the classification in [12], a class that can be expanded in terms of harmonic sums, as we will show now.

Using $(\epsilon)_i = -\sum_{n=1}^{\infty} \left(-\frac{\epsilon}{i}\right)^n (1+\epsilon)_i$ for two terms in the numerator, $(a\epsilon)_i$ and $(6\epsilon)_i$, and solving one of the two new sums, the above sum reads

$$\sum_{n=2}^{\infty} \frac{(-6\epsilon)^n a}{6-a} \left(1 - \left(\frac{a}{6}\right)^{n-1}\right) \sum_{i=1}^{\infty} \frac{(\frac{1}{2} + \epsilon)_i (1+6\epsilon)_i}{i^n (\frac{1}{2} + 5\epsilon)_i (1)_i}.$$

²Note that both generalized hypergeometric functions we need can be conveniently summarized introducing only one parameter, $a = 2$ and $a = 4$, respectively.

The next step is to expand the Pochhammer symbols in ϵ , using [12]

$$\begin{aligned} \left(\frac{1}{2} + \epsilon\right)_i &= \\ &= \left(\frac{1}{2}\right)_i \exp\left(-\sum_{k=1}^{\infty} \frac{(-2\epsilon)^k}{2k} [S_k(2i) - S_{-k}(2i)]\right) \\ (1+\epsilon)_i &= \\ &= (1)_i \exp\left(-\sum_{k=1}^{\infty} \frac{(-2\epsilon)^k}{2k} [S_k(2i) + S_{-k}(2i)]\right). \end{aligned}$$

It becomes clear now why the sum had to be *balanced*: the Pochhammer symbols $(\frac{1}{2})_i$ and $(1)_i$ cancel between numerator and denominator. Expanding the exponentials and rewriting products of S-sums as single S-sums leaves us with sums of generic form $\sum_i S_{\vec{k}}(2i)/i^n$ to be solved.

To this end, we use the trick employed in [12], introducing a delta function on the integers:

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{S_{\vec{k}}(2i)}{i^n} &= 2^n \sum_{i=1}^{\infty} \frac{1}{(2i)^n} S_{\vec{k}}(2i) \\ &= \frac{2^n}{2} \sum_{j=1}^{\infty} \frac{1^j + (-1)^j}{j^n} S_{\vec{k}}(j) \\ &= \frac{2^n}{2} [S_{n, \vec{k}}(\infty) + S_{-n, \vec{k}}(\infty)]. \end{aligned} \quad (17)$$

Putting it all together, and doing some trivial algebra with Gamma functions,

$$\begin{aligned} T_3 &= 3 \frac{2^{6\epsilon}}{16\epsilon} (1-2\epsilon)^3 \frac{\Gamma^2(1+\epsilon)}{\Gamma(1-\epsilon)} \frac{\Gamma(\frac{1}{2}+3\epsilon)\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(\frac{1}{2}+5\epsilon)\Gamma(\frac{1}{2})} \times \\ &\times \sum_{n=0}^{\infty} (-4\epsilon)^n (1+3^{n+2}-2^{n+3}) \sum_{j=1}^{\infty} \frac{1^j + (-1)^j}{j^{n+2}} \times \\ &\times \exp\left(\sum_{k=1}^{\infty} \frac{(-2\epsilon)^k}{(-2k)} [(6^k - 5^k + 1)S_k(j) + \right. \\ &\left. + (6^k + 5^k - 1)S_{-k}(j)]\right). \end{aligned} \quad (18)$$

For expanding the Gamma functions around integer and half-integer arguments, we use

$$\Gamma(1+\epsilon) = \exp\left(-\epsilon\gamma + \sum_{n \geq 2} (-\epsilon)^n \frac{\zeta(n)}{n}\right), \quad (19)$$

$$\Gamma\left(\frac{1}{2} + \epsilon\right) = \sqrt{\pi} \exp\left(-\epsilon(\gamma + 2 \ln 2) + \sum_{n \geq 2} (-\epsilon)^n \frac{\zeta(n)}{n} (2^n - 1)\right), \quad (20)$$

such that Eq. (18) gives

$$\begin{aligned} T_3 = & +\epsilon^{-1} \left(+\frac{3}{16} \zeta_2 \right) \\ & + \left(-\frac{9}{8} \zeta_2 + \frac{9}{4} \zeta_2 \ln 2 - \frac{21}{8} \zeta_3 \right) \\ & + \epsilon \left(+\frac{3}{2} \ln^4 2 + \frac{9}{4} \zeta_2 - \frac{27}{2} \zeta_2 \ln 2 + \frac{9}{2} \zeta_2 \ln^2 2 \right. \\ & \quad \left. - \frac{207}{40} \zeta_2^2 + \frac{63}{4} \zeta_3 + 36 a_4 \right) \\ & + \epsilon^2 \left(-9 \ln^4 2 + \frac{18}{5} \ln^5 2 - \frac{3}{2} \zeta_2 + 27 \zeta_2 \ln 2 \right. \\ & \quad \left. - 27 \zeta_2 \ln^2 2 + 18 \zeta_2 \ln^3 2 + \frac{621}{20} \zeta_2^2 \right. \\ & \quad \left. - \frac{621}{10} \zeta_2^2 \ln 2 - \frac{63}{2} \zeta_3 - \frac{87}{8} \zeta_3 \zeta_2 + \frac{4743}{16} \zeta_5 \right. \\ & \quad \left. - 216 a_4 - 432 a_5 \right) + \mathcal{O}(\epsilon^3), \quad (21) \end{aligned}$$

where $a_n = \text{Li}_n(1/2)$ are polylogarithms. Due to space limitations, we have not shown all known coefficients above. For the full result including $\mathcal{O}(\epsilon^6)$ (whose coefficient entails weight-9 numbers like s_{9a}), see [13]. To 30 digit accuracy, we have

$$\begin{aligned} T_3 = & + 0.308425137534042456838577843746 \epsilon^{-1} \\ & - 2.44054202205177103588797505371 \\ & + 15.7704017490035557236803526639 \epsilon \\ & - 100.504895668775469387159215590 \epsilon^2 \\ & + 625.061590652357317030007023983 \epsilon^3 \\ & - 3839.50600368202197971416339398 \epsilon^4 \\ & + 23392.0898313583063340598037114 \epsilon^5 \\ & - 141772.262976662567133066547299 \epsilon^6 \\ & + \mathcal{O}(\epsilon^7). \end{aligned}$$

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